Family of solvable generalized random-matrix ensembles with unitary symmetry

K. A. Muttalib

Department of Physics, University of Florida, P.O. Box 118440, Gainesville, Florida 32611-8440, USA

J. R. Klauder

Department of Physics, University of Florida, P.O. Box 118440, Gainesville, Florida 32611-8440, USA
and Department of Mathematics, University of Florida, P.O. Box 118105, Gainesville, Florida 32611-8105, USA
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We construct a very general family of characteristic functions describing random matrix ensembles (RME) having a global unitary invariance, and containing an arbitrary, one-variable probability measure, which we characterize by a "spread function." Various choices of the spread function lead to a variety of possible generalized RMEs, which show deviations from the well-known Gaussian RME originally proposed by Wigner. We obtain the correlation functions of such generalized ensembles exactly and show examples of how particular choices of the spread function can describe ensembles with arbitrary eigenvalue densities as well as critical ensembles with multifractality.

The concept of random matrix ensembles (RME) proposed by Wigner to describe statistical properties of the eigenvalues and eigenfunctions of complex nuclei [1], has proved to be a very useful idea in the studies of a wide variety of physical systems including the equilibrium and transport properties of disordered quantum systems, quantum chaos, two-dimensional quantum gravity, conformal field theory, and chiral phase transitions in quantum chromodynamics, as well as financial correlations and wireless communications [2,3]. The underlying reason for such a wide range of applications is the universality of the correlations between the eigenvalues of given classes of RMEs. For example, once appropriate variables are chosen in which the mean spacing between eigenvalues is unity, the Gaussian ensembles of random $N \times N$ matrices (with given symmetries) have, in the $N \to \infty$ limit, a universal (zero parameter) form for the nearest-neighbor spacing distribution or the spectral rigidity (number variance of eigenvalues in a given range) known generally as the Wigner-Dyson distribution [1]. The appropriately scaled energy levels of complex nuclei and transmission levels of weakly disordered mesoscopic metals both follow the above universal distributions even though the physical sizes of the systems differ by about nine orders of magnitude [4].

More recently, much interest has been generated in finding RMEs that *deviate* from the universal properties of the Gaussian ensembles in specific ways. One particular example is the attempt to find "critical" ensembles relevant for systems at the critical point of, e.g., the Anderson transition in disordered conductors where the spacing distribution or the spectral rigidity is known to deviate from those of Gaussian RMEs [5–7]. Another example is the attempt to find a one-parameter generalization of the Gaussian RME that can describe a monotonic change from the universal Wigner-Dyson distribution to a completely uncorrelated Poisson distribution as the parameter is changed, which may be relevant for a crossover from a chaotic to an integrable system [8]. Yet another example is the observation for financial cross-correlation matrices in which statistics of most eigenvalues

agree with the universal predictions of Gaussian RMEs but there are deviations for a few of the largest eigenvalues [9]. It is therefore of great interest to a wide variety of areas and disciplines to study RMEs that are in some sense generalizations of the Gaussian RMEs.

In seeking generalizations of Gaussian RMEs, suitable for arbitrarily many variables, it is natural to begin with the probability distributions $P_N(X)$, where X denotes an $N \times N$ Hermitian matrix. Gaussian ensembles have centered distributions that are in fact (exponential) functions of the single variable $\text{tr}(X^2)$. It is natural to restrict our generalizations to probability distributions $P_N(X) = W_N[\text{tr}(X^2)]$. One such example that might come to mind could be $P_N(X) \propto \exp\{-[\text{tr}(X^2)]^2\}$, but although such a proposal is satisfactory for any *finite* N, it fails to generalize to a valid new distribution as $N \to \infty$. The clue to discovering the proper class of generalizations is to work not directly with the distribution $P_N(X)$ themselves, but with their Fourier transforms, i.e., with the associated *characteristic functions* [11] $C_N(T)$ given by

$$C_N(T) = \int e^{i\text{tr}(TX)} P_N(X) dV_X, \qquad (1)$$

with $C_N(0) = 1$. The integration is over the invariant Haar measure that preserves hermiticity. In the present paper we first prove that if $C_N(T)$ is a function of $\text{tr}(T^2)$ only, then the *most general* $C_N(T)$, valid for arbitrarily large N, can always be written as

$$C_N(T) = \int_0^\infty e^{-b\text{tr}(T^2)} f(b) db, \quad \int_0^\infty f(b) db = 1, \qquad (2)$$

where $f(b)$ is *any* non-negative function, which may be chosen phenomenologically. While Eq. (2) is evidently a positive superposition of Gaussians, a vast family of distributions is thereby included, e.g., Cauchy, Levy, etc.; this result generalizes the specialized and more restrictive examples of

[12]. We then show that the n-point correlation function for the corresponding unitary RME [10] of the matrices X with eigenvalues x_i can be written down *exactly* as

$$R_n(x_1, \ldots, x_n) = \int_0^\infty \frac{dbf(b)}{(4b)^{n/2}} \det[K_N^G(\bar{x}_i, \bar{x}_j)]_{i,j=1,2,\ldots,n}. \quad (3)$$

Here we have defined $\bar{x}_i = x_i/2\sqrt{b}$, and $K_N^G(x_i, x_j) = \sum_{n=0}^{N-1} \varphi_n(x_i)\varphi_n(x_j)$ is the well-known two-point kernel of the Gaussian RME [1], where $\varphi_n(x) \propto e^{-x^2/2}H_n(x)$ are orthonormal functions associated with the Hermite polynomials $H_n(x)$. In particular, the one-point function is just the density of levels, given (in the large N limit) by

$$\sigma_N(x) = \frac{\sqrt{2N}}{2\pi} \int_{x^2/8N}^\infty \frac{dbf(b)}{b^{1/2}} \sqrt{1 - x^2/8Nb}, \quad (4)$$

where we have used the known result that for large N, $K_N^G(x,x) \equiv \sigma_N^G(x) = \sqrt{2N-x^2}/\pi$ for $|x| < \sqrt{2N}$ and zero otherwise [1]. It is clear that one can obtain different x and N dependence for the densities by choosing different $f(b)$. Similarly, the two-point cluster function defined as $T_2(x_1, x_2) \equiv -R_2(x_1, x_2) + R_1(x_1)R_1(x_2)$ has the form $T_2 = T_2^0 - \delta T_2$, where

$$T_2^0(x_1, x_2) = \int_0^\infty \frac{dbf(b)}{4b}[K_N^G(\bar{x}_1, \bar{x}_2)]^2 \quad (5)$$

and

$$\delta T_2 = \int_0^\infty \frac{dbf(b)}{4b} \sigma_N^G(\bar{x}_1)\sigma_N^G(\bar{x}_2) - \sigma_N(x_1)\sigma_N(x_2). \quad (6)$$

We call $f(b)$ the spread function. Note that the Gaussian RME corresponds to the choice $f(b) = \delta(b - b_0)$, for which δT_2 is identically zero. Other choices of the spread function can describe a variety of possible generalized RMEs. Alternatively, correlation functions of a physically relevant RME of matrices X characterized by a given $C_N(T)$ can be obtained exactly if the corresponding spread function $f(b)$ can be identified. For example, $C_N(T) = e^{-b_0\sqrt{\operatorname{tr} T^2}}$ will correspond to a choice of $f(b) \propto b^{-3/2}e^{-b_0^2/b}$, for which the exact correlation functions can be easily written down.

The universal features of Gaussian RMEs arise in the $N \to \infty$ limit and when the variables are chosen in which the mean level spacing is unity (this is known as unfolding). In place of x_1 and x_2 one defines new variables ρ and ζ such that $d\rho = \sigma(x_1)dx_1$, $d\zeta = \sigma(x_2)dx_2$, and the new cluster function

$$Y_2(\rho, \zeta)d\rho d\zeta \equiv T_2(x_1, x_2)dx_1 dx_2 \quad (7)$$

is well defined everywhere in the limit $N \to \infty$. For Gaussian RMEs, there exists the sum rule that the integral of $Y_2(\rho, \zeta)$ over ρ is always unity. It has been argued [13,14] that the violation of this sum rule is a signature of critical ensembles, where the deficit of the sum rule (for translationally invariant cluster functions)

$$\eta \equiv 1 - \int_{-\infty}^\infty Y_2(\rho, \zeta)d\rho \quad (8)$$

is related to the multifractality of wave functions at the critical point [15]. We will show that there are choices for $f(b)$ for which the sum rule is violated. These choices would then correspond to critical ensembles.

We begin by proving that if $C(T)$ is a function of $\operatorname{tr}(T^2)$, then the most general $C_N(T)$, valid for arbitrarily large N, can be written in the form Eq. (2). Note that if $C(T)$ is a function of $\operatorname{tr}(T^2)$ only, then from Eq. (1) we have $P_N(U^\dagger XU) = P_N(X)$ where U is unitary. Thus the distribution is invariant under a rotation of basis.

Proof of Eq. (2). We suppose that

$$C(T) \equiv E(\|T\|_2^2) \equiv \int e^{i\operatorname{tr}(TX)}d\mu(X), \quad (9)$$

for all Hermitian T with $\|T\|_2^2 \equiv \operatorname{tr}(T^2) < \infty$, where μ is a suitable probability measure. It follows that $C(T-S) = E(\|T - S\|_2^2)$ is a real, continuous function of positive type, and the Gel'fand-Naimark-Segal (GNS) Theorem [16] ensures that there exist vectors $|T\rangle$ and $|S\rangle$ in a separable Hilbert space such that

$$\langle S|T\rangle \equiv E(\|T - S\|_2^2). \quad (10)$$

A family of operators $V(U)$ may be defined by

$$\langle S|V(U)|T\rangle \equiv \langle S|T + U\rangle = E(\|T + U - S\|_2^2), \quad (11)$$

and it readily follows that $V(U)$ is an Abelian group of unitary operators for which the operator norm $\|V(U)\| = 1$. Now consider the sequence $U^{(M)} = \{U_{rs}^{(M)}\}$ where

$$U_{rs}^{(M)} \equiv \sqrt{u}\delta_{rM}\delta_{sM}, \quad u \geq 0, \quad 1 \leq M < \infty. \quad (12)$$

It follows that the weak operator limit

$$\lim_{M \to \infty} \langle S|V(U^{(M)})|T\rangle = E(\|T - S\|_2^2 + u) \equiv \langle S|A(u)|T\rangle, \quad (13)$$

an expression that defines the operator $A(u)$ for all $u \geq 0$. Clearly, $A(u)^\dagger = A(u)$, $A(u)A(v) = A(u+v)$, and $\|A(u)\| \leq 1$, and therefore $A(u) = e^{-uB}$, where $B^\dagger = B \geq 0$. Hence,

$$E(u) = \langle 0|e^{-uB}|0\rangle = \int_0^\infty e^{-ub}dm(b), \quad (14)$$

where m is a probability measure, and the latter relation follows from the spectral representation for B. Finally, if we assume that m is absolutely continuous and replace u by $\|T\|_2^2$, we recover Eq. (2). This completes the proof of Eq. (2).

We next show that given Eq. (2), the n-point correlation function is given by Eq. (3).

Proof of Eq. (3). Given the characteristic function Eq. (2), the probability density is, from Eq. (1),

$$P_N(X) \propto \int_0^\infty dbf(b) \int e^{-i\operatorname{tr}(TX)}e^{-b\operatorname{tr}(T^2)}dV_T. \quad (15)$$

The integral over N^2 independent elements of T results in

$$P_N(X) \propto \int_0^\infty \frac{db f(b)}{b^{N^2/2}} e^{-\mathrm{tr}(X^2)/4b}. \tag{16}$$

The joint probability distribution of the eigenvalues of X is then given by [1]

$$P_N(\{x_i\}) \propto \int_0^\infty \frac{db f(b)}{b^{N^2/2}} \prod_{i<j} (x_i - x_j)^2 e^{-\Sigma_i x_i^2/4b}, \tag{17}$$

where $\Pi_{i<j}(x_i - x_j)^2$ is just the Jacobian of transformation from the matrix element to the eigenvalue and eigenvector coordinates. We now take advantage of the known results for Gaussian random matrix ensembles by noting that before the b integral, a change of variables $\bar{x}_i = x_i/2\sqrt{b}$ changes the distribution to exactly Gaussian RME with some additional b-dependent terms; this leads directly to Eq. (3). This completes the proof of Eq. (3).

As an example, let us consider the spread function $f(b)$ $\propto b^{N^2/2+\nu-1} e^{-\varepsilon^2/4b} e^{-\gamma b}$. Then

$$P_N(X) \propto \left(\frac{\varepsilon^2 + \mathrm{tr}(X^2)}{4\gamma} \right)^{\nu/2} K_\nu(\sqrt{\gamma[\varepsilon^2 + \mathrm{tr}(X^2)]}), \tag{18}$$

where K_ν is a modified Bessel function. In the limit $\gamma \to 0$, choices of ν would include ensembles of Lévy matrices [17]. If $\nu = -n$ where n is a positive integer, then in the same limit $\gamma \to 0$ this gives rise to $P_N(X) \propto [\varepsilon^2 + \mathrm{tr}(X^2)]^{-n}$, while in the opposite limit $\gamma \to \infty$ we get $P_N(X) \propto [\varepsilon^2 + \mathrm{tr}(X^2)]^{-(n/2+1/4)} e^{-\sqrt{\gamma[\varepsilon^2 + \mathrm{tr}(X^2)]}}$. The two-point correlation function for all γ and ε can be written down exactly from Eqs. (5) and (6).

As an explicit example of how different behavior of the correlation functions may arise for some choices of $f(b)$ in the large N limit, let us consider the level density for

$$f(b) = c_N \frac{2\bar{b} + 1}{[\bar{b}(\bar{b}+1)]^{3/2}}, \quad b < b_0;$$
$$= 0, \quad b > b_0, \tag{19}$$

where $\bar{b} \equiv \sqrt{8Nb}$ and b_0 is determined from the normalization condition. We choose c_N such that $\bar{b}_0 \equiv \sqrt{8Nb_0} \gg 1$. Then the density obtained from Eq. (4) is given by

$$\sigma_N(x) \approx \frac{1}{\pi} \frac{c_N}{\sqrt{x(x+1)}}, \tag{20}$$

which is similar to that satisfied by the transmission eigenvalues in disordered conductors and is known to lead to deviations from Gaussian RME [4,7]. Note that the density diverges as $1/\sqrt{x}$ in the limit $x \to 0$. It is clear that by appropriately choosing the spread function $f(b)$, one can obtain a variety of densities that can go to zero, a constant or infinity as a function of x as $x \to 0$. One can also choose $f(b)$ to obtain an N-dependent density at the origin that goes to zero, a constant or infinity in the $N \to \infty$ limit. An N-independent finite density at the origin was conjectured to be important for the properties of the q random matrix ensembles reviewed in [18].

In particular, $f(b)$ can be defined such that in the limit

$N \to \infty$ both $\sigma(x)$ and $\delta T_2(x,y)$ become constant. In such cases if we define

$$\phi(\lambda) = \int_0^\infty db f(b) e^{-\lambda/\sqrt{b}}, \tag{21}$$

then $\phi(0) = 1$, $\sigma(x) = \sigma_0 = -(\sqrt{2N}/2\pi)\phi'(0)$ and $\delta T_2(x,y) = (N/2\pi^2)\{\phi''(0) - [\phi'(0)]^2\}$. We now choose $\rho = \sigma_0 x$, $\zeta = \sigma_0 y$ to scale the density to unity. Then the scaled cluster function defined in Eq. (7) has the simple form $Y_2(\rho,\zeta) = T_2(x_1,x_2)/\sigma_0^2$. Using Eqs. (5) and (6) to define Y_2^0 and δY_2, we find

$$Y_2^0(\rho,\zeta) = \frac{1}{\pi^2} \int_0^\infty \frac{db f(b)}{(\rho-\zeta)^2} \sin^2\left(\sqrt{\frac{N}{2b}} \frac{(\rho-\zeta)}{\sigma_0} \right), \tag{22}$$

where we have used the known large N behavior for $K_N^G(x,y)$ [1], and

$$\delta Y_2(\rho,\zeta) = \frac{\phi''(0)}{[\phi'(0)]^2} - 1. \tag{23}$$

The deficit of the sum rule Eq. (8) then takes the form

$$\eta = 1 - \int_{-N/2}^{N/2} d\rho Y_2^0(\rho,\zeta) + N\left[\frac{\phi''(0)}{[\phi'(0)]^2} - 1 \right]. \tag{24}$$

If $\sigma_0 \propto \sqrt{N}$, the argument of the sine function in Y_2^0 is independent of N. Then the integral of Y_2^0 is just unity for all $f(b)$ in the limit $N \to \infty$. Using the inequality

$$\int db f(b) b^{-1/2} \leq \sqrt{\int db f(b) b^{-1} \int db f(b)} \tag{25}$$

and the normalization of $f(b)$, we obtain $\eta \geq 0$. The equality sign holds only if $f(b) \propto \delta(b - b_0)$, which is the Gaussian RME. As an explicit example of finite positive η in the large N limit, let us choose

$$f(b) \propto b^\alpha e^{-\beta/\sqrt{b}} e^{-\gamma\sqrt{b}}. \tag{26}$$

With $\alpha = -3/4$, this immediately leads to $\eta = N/2\sqrt{\beta\gamma}$. If $\sqrt{\beta\gamma} \propto N$, we get a well-defined critical ensemble in the thermodynamic limit. Note that for finite η, Eq. (23) gives $\delta Y_2 = \eta/N \to 0$ in the $N \to \infty$ limit.

The variance $\Sigma(s) = \langle n^2 \rangle - \langle n \rangle^2$ of the number of eigenvalues n in an interval $(-s/2, s/2)$ is given by

$$\Sigma(s) = \int_{-s/2}^{s/2} d\rho \int_{-s/2}^{s/2} d\zeta [\delta(\rho-\zeta) - Y_2(\rho-\zeta)]. \tag{27}$$

For Gaussian RME, $\Sigma(s) \propto \ln s$ for large s. For critical ensembles discussed in the literature [5,7], $\Sigma(s) \propto s$. Our choice of the spread function $f(b)$ in Eq. (26), which gives rise to a constant δY_2, produces a term $\Sigma(s) \propto s^2$ in addition to terms $\Sigma(s) \propto \ln s$ from Y_2^0. However, as mentioned earlier, this term is proportional to η/N, which vanishes in the large N limit for finite η. Equation (26) therefore corresponds to a kind of critical ensemble with $\Sigma(s) \propto \ln s$ for large s.

Note from Eq. (5) that in general δT_2 is not translationally invariant; clearly the critical ensembles in such cases will not

give rise to number variance $\Sigma(s) \propto \ln s$, and will correspond to a different class. We also get a different class of critical ensembles if the contribution to η from Y_2^0 is different from unity. For choices of $f(b)$ which diverge as $b \to 0$, changing the order of the integrals over ρ and b in Eq. (8) may not be justified, and the integral in Eq. (22) may depend on the choice of such $f(b)$.

In summary, we have constructed a generalized random matrix ensemble whose characteristic function contains an arbitrary non-negative spread function $f(b)$ with the only condition that $\int_0^\infty f(b)db = 1$. The correlation functions of the generalized ensembles are exactly solvable for any given $f(b)$. Various choices of $f(b)$ lead to a variety of possible density of levels $\sigma_N(x)$, which can depend on x or N in a variety of different ways, leading to possible deviations from Gaussian Ensembles. In particular, we showed that it is possible to choose forms of $f(b)$ that lead to violations of the sum rule for the scaled two-point cluster function $Y_2(\rho, \zeta)$ where the deficit of the sum rule η, as given in Eq. (8), is a characteristic of critical ensembles with multifractal wave functions. Unlike critical ensembles discussed in the literature, these sum rule violations can correspond to different forms for the number variance $\Sigma(s)$, corresponding to different classes of critical ensembles. These solvable generalized ensembles should therefore be of interest in a wide range of areas where random matrix ensembles play an important role. While there are only a few known examples of generalized RMEs for which correlation functions can be evaluated exactly [6,7], a given model of $C_N(T)$ or $P_N(X)$ characterizing a physically relevant generalized RME becomes exactly solvable if the corresponding spread function $f(b)$ can be found. This opens up the possibility to obtain exact results for a variety of interesting and physically useful generalized RMEs.

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